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Energy transmission by surface waves through an opening

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In an ocean of uniform depth the propagation of small-amplitude plane waves is impeded by two vertical semi-infinite perfectly reflecting barriers extending from the bottom of the sea to above the free surface. The two screens do not lie in general in the same plane, and they are separated by a gap through which wave energy is transmitted from the open sea into the sheltered region. A transmission coefficient is established for small gap widths relative to the wavelength and the agreement with existing theoretical results for special cases is found to be very good.

1. Introduction

The transmission of two-dimensional plane waves through an aperture defined by the vertical edges of two semi-infinite planes is examined, with the objective of determining the amount of wave energy that penetrates into the sheltered region. The geometry of the problem is shown in figure 1.

The waves are assumed to be surface water waves and the planes to extend to the horizontal bottom of the sea. The assumption regarding the kind of waves is not restrictive and is introduced for ease of visualization.

The planes form an exterior angle $2\pi\beta$ towards the area representing the open sea. The incident wave train is monochromatic and of small amplitude described by the conventional first-order linear theory. Two configurations are investigated: (*a*) the symmetrical case where the gap between the two planes is AB; and (*b*) the asymmetrical one in which the gap is OA (figure 1).

The technique of matched asymptotic expansions is employed in the long-wave asymptotic limit $\epsilon = kd \to 0$, k the wavenumber, d the gap width. The inner solution is based on Lamb's (1932) argument that in the two-dimensional problem of waves passing through an aperture and in the immediate neighbourhood of the opening the motion 'must resemble the flow of a liquid through the same aperture' and an approximation is obtained by comparison with the results of the theory of the steady two-dimensional fluid motion as developed by the use of conformal transformations.

The outer solution can be derived from the radiation of a line source located at the apex O and the scattered field of plane wave train incident on a wedge formed by the two semi-infinite planes extended to meet at the point O (figure 1). The solution of the diffraction of waves by a wedge is known (Oberhettinger 1958).

The general problem of wave energy transmission through apertures has been the subject of numerous investigations; however most of them have been concerned with

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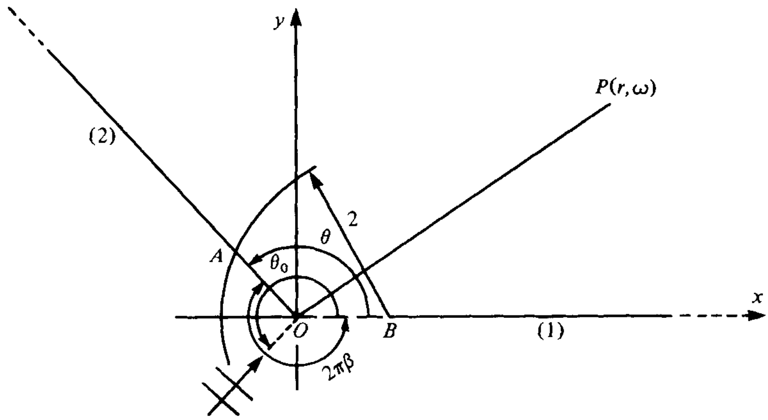

FIGURE 1. Geometry of the problem.

horizontal rather than vertical slots. A survey of problems of flow through small holes, involving matching techniques, has been given by Tuck (1975). A case similar to the one examined in this paper has been treated by Liu (1975), who tackled by matched expansions the scattering of water waves by a pair of parallel semi-infinite barriers. It is noted that this configuration is not a special case of our problem since for $\beta = 1$ the two planes coincide.

For both geometries stated above simple expressions for the transmission factor are found. The parameters of the results are the angle θ between the two barriers and the non-dimensional gap width $\delta = d/\lambda$, λ being the incident wavelength. For the case $\beta = \frac{1}{2}$ Lamb's result (Lamb 1932, art. 305) is reproduced.

2. The outer problem

Open sea

The problem involves two independent length scales d and λ, therefore it is seen as a singular perturbation problem, and a single asymptotic solution cannot be found valid throughout the flow field. As $\epsilon \to 0$ it is clear that the outer field of the ocean region ($\theta < \omega < 2\pi$) will tend to the known solution f_w of diffraction by a wedge, which suggests an expansion for the outer approximation

$$f(x,y) \sim f_w(x,y) + \sum_n g_n(\epsilon) f_n(x,y), \tag{2.1}$$

where $g_n(\epsilon) \to 0$ as $\epsilon \to 0$, and the incident plane wave has been incorporated in f_w.

Some of the functions f_n will contain the behaviour of a line source at the origin, because at large distance the flow resembles that caused by a negative line source located at the opening of the two plane breakwaters. This behaviour is expressed by

$$-\tfrac{1}{4} i H_0(kr), \tag{2.2}$$

where H is the Hankel function of the first kind. It should be noted that when ϵ is small enough but not zero the 'source field' becomes stronger than the 'wedge field', whereas this latter dominates when $\epsilon = 0$.

An expression of the solution of the reduced wave equation for the diffraction of plane waves by a wedge is

$$f_w(r,\omega;\theta_0) = \frac{2\pi}{\theta} \sum_{m=0}^{\infty} \epsilon_m J_\mu(kr) \cos\mu\omega \cos\mu\theta_0 \,.\, e^{-\frac{1}{2}i\mu\pi}, \tag{2.3}$$

where θ_0 is the angle of incidence, $\epsilon_m = 1$ for $m = 0$, $\epsilon_m = 2$ for $m \geqslant 1$, and $\mu = m\pi/\theta$ (Felsen & Marcuvitz 1973).

For $kr \ll 1$ a small-argument expansion of the Bessel functions in equation (2.3) gives for $\theta < \pi$

$$f_w \sim \frac{1}{\beta} + \frac{4}{\Gamma(1/2\beta)} (\tfrac{1}{2}kr)^{1/2\beta} \exp(\tfrac{1}{4}i\pi/\beta) \cos\frac{\omega}{2\beta} \cos\frac{\theta_0}{2\beta} + O[(kr)^{1/\beta}], \tag{2.4}$$

where $\Gamma(x)$ denotes the gamma function.

The relation (2.2) is true in free space. In the present context a sector of angle θ is assumed occupied; therefore the formula (2.2) is divided by β and becomes

$$-\frac{i}{4\beta} H_0(kr). \tag{2.5}$$

Its expansion is

$$-\frac{i}{4\beta} - \frac{1}{2\pi\beta}(\ln 2 - \gamma) + \frac{1}{2\pi\beta}\ln(kr), \quad kr \to 0, \tag{2.6}$$

where γ is Euler's constant.

Sheltered region

For an observer far away in the lee of the barriers the flow appears to be produced by a line source at the apex. Consequently expressions corresponding to formulae (2.5), (2.6) and of opposite sign will hold:

$$+\frac{i}{4(1-\beta)} H_0(kr); \tag{2.7}$$

$$+\frac{i}{4(1-\beta)} + \frac{1}{2\pi(1-\beta)}(\ln 2 - \gamma) - \frac{1}{2\pi(1-\beta)}\ln(kr), \quad kr \to 0. \tag{2.8}$$

3. The inner problem

Symmetrical opening

It was stated previously that the basic inner solution can be derived from the corresponding two-dimensional fluid motion. The case with a symmetrical opening in which $OA = OB$ has been solved by Harris (1901). By a rotation of the co-ordinate system through an angle $\pi\beta$ we get x and y in terms of the potential f of the steady state and the stream function p:

$$x = \frac{\beta^\beta(1-\beta)^{1-\beta}}{\sin(1-\beta)\pi}[e^{\beta f}\cos\beta(p-\pi) + e^{(\beta-1)f}\cos(1-\beta)(p-\pi)]; \tag{3.1}$$

$$y = \frac{\beta^\beta(1-\beta)^{1-\beta}}{\sin(1-\beta)\pi}[e^{\beta f}\sin\beta(p-\pi) - e^{(\beta-1)f}\sin(1-\beta)(p-\pi)]. \tag{3.2}$$

An important point in this analysis is that the length of the segment OB depends on β, having a value $(OB) = (OA) = \operatorname{cosec}(1-\beta)\pi$, so that $(AB) = 2$. This limitation is relaxed later on. Writing equations (3.1) and (3.2) in compact form with $z = x+jy$, $w = f+jp$ we derive

$$z = \frac{\beta^\beta(1-\beta)^{1-\beta}}{\sin(1-\beta)\pi} e^{\beta(w-j\pi)}(1-e^{-w}), \tag{3.3}$$

where j is the imaginary unit in the complex planes of figure 2.

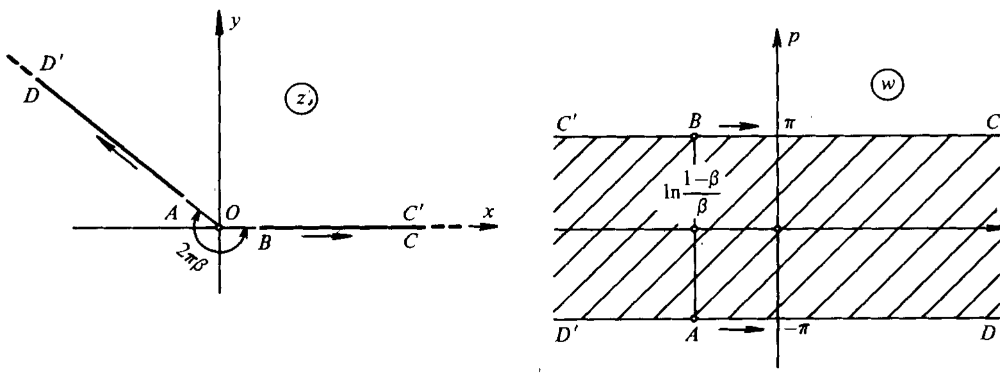

FIGURE 2. The mapping of equation (3.3).

It is required that the function w is expressed in terms of z, in order that f, which is needed for the matching procedure, can be obtained as $f = R(w)$. To invert equation (3.3) we construct the mapping that it represents in figure 2.

Putting $p = \pi$ in the above equation the line

$$z = \frac{\beta^\beta(1-\beta)^{1-\beta}}{\sin(1-\beta)} e^{\beta f}(1+e^{-f})$$

is found, which has a minimum at the point B where

$$\frac{d}{df}[e^{\beta f} + e^{(\beta-1)f}] = 0,$$

whence

$$f_B = \ln\frac{1-\beta}{\beta}.$$

From figure 2 it is deduced that for the open sea sector $R(w) \to \infty$; therefore, as a first approximation the inverse of equation (3.3) is written as

$$w \sim -\frac{1}{\beta}\ln B(\beta) + j\pi + \frac{1}{\beta}\ln z,$$

where $$B(\beta) = \beta^\beta(1-\beta)^{1-\beta}/\sin(1-\beta)\pi.$$

Taking the real parts we obtain

$$f \sim -\frac{1}{\beta}\ln B(\beta) + \frac{1}{\beta}\ln r, \quad r \to \infty. \tag{3.4}$$

For the sheltered sector, $R(w) \to -\infty$ and therefore ignoring the first term in equation (3.3) we find to the first order

$$w \sim \frac{1}{1-\beta}\ln A - \frac{\pi j}{\beta-1} + \frac{1}{\beta-1}\ln z = \frac{1}{1-\beta}\ln B(\beta) + \pi j + \frac{1}{\beta-1}\ln z$$

and taking the real parts

$$f \sim \frac{1}{1-\beta}\ln B(\beta) - \frac{1}{1-\beta}\ln r, \quad r \to \infty. \tag{3.5}$$

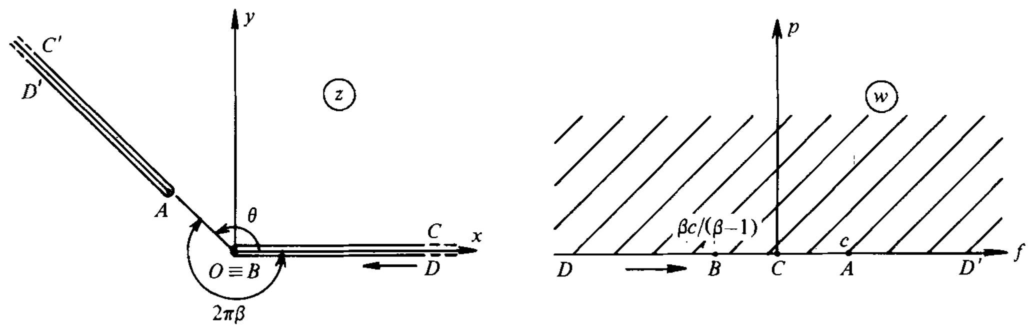

FIGURE 3. The mapping of equation (3.7).

Asymmetrical opening

In this case the opening is defined by OA rather than AB. For the general case where $(OA) \neq (OB)$ it is found through a Schwarz–Christoffel transformation (Kober 1957) that

$$z = \left[\frac{1}{2\beta}w^{2\beta}+\frac{a-c}{2\beta-1}w^{2\beta-1}+\frac{ac}{2(1-\beta)}w^{2\beta-2}\right]e^{-2\pi j\beta}, \qquad (3.6)$$

where the positive constants a and c depend on the position of point B and on the gap width.

Relation (3.6) maps the whole z plane with two semi-infinite cuts to the half-plane $p > 0$ and gives for the point B the co-ordinates

$$x_B = (-a)^{2\beta-2}\left[\frac{a^2}{2\beta}-\frac{a-c}{2\beta-1}a+\frac{ac}{2(1-\beta)}\right], \quad y_B = 0.$$

It is required that point B coincides with O; this gives the value of a as $\beta c/(1-\beta)$ and equation (3.6) becomes

$$z = \left[\frac{1}{2\beta}w^{2\beta}+\frac{c}{1-\beta}w^{2\beta-1}+\frac{\beta c^2}{2(1-\beta)^2}w^{2\beta-2}\right]e^{-2\pi j\beta}. \qquad (3.7)$$

The mapping described by this equation is shown in figure 3.

The point A has the co-ordinates $z_A = 2c^{2\beta}\beta(1-\beta)^2 e^{-2\pi j\beta}$ and putting $|z_A| = \delta$ we derive for c the value

$$c = [2\delta\beta(1-\beta)^2]^{1/2\beta}. \qquad (3.8)$$

It can be seen from figure 3 that for the ocean sector $|w| \gg 1$ for $|z| \to \infty$. Therefore to the first order

$$w \sim -(2\beta z)^{1/2\beta}.$$

Putting this value of w into equation (3.7), we obtain to the second order

$$w \sim -(2\beta z)^{1/2\beta} - \frac{c}{1-\beta}. \qquad (3.9)$$

The parameter c brings to this order information about the gap width.

For the sheltered sector of the plane (x, y) it is noted from figure 3 that, for $|z| \to \infty$, $|w| \ll 1$. Therefore the terms of equation (3.7) are treated in the reverse order of magnitude than previously. After successive approximation and some algebra the following expression is found:

$$w \sim (\alpha z)^{1/2(\beta-1)} t + N(\alpha z)^{1/(\beta-1)} t^2 + \ldots, \quad |z| \to \infty, \tag{3.10}$$

where $$\alpha = \frac{2}{\beta}\left(\frac{1-\beta}{c}\right)^2, \quad t = \exp\left(\frac{\pi j \beta}{\beta-1}\right), \quad N = \frac{\beta c^3}{4(1-\beta)^4}.$$

4. Matching

Having the inner and outer basic approximations at our disposal we can proceed to the matching of their expansions. Inspection of equations (3.4), (3.9) and of results in similar problems (Liu 1975) suggests that the source effect is more pronounced than the wedge effect in the early stages of the matching. So, for the outer solution we have the following.

(i) *Ocean sector:*

$$f \sim f_w - \frac{iQ}{4\beta} H_0(kr), \tag{4.1}$$

where Q is a constant representing the strength of the line source to be determined by matching.

(ii) *Sheltered sector:*

$$f \sim \frac{iQ}{4(1-\beta)} H_0(kr). \tag{4.2}$$

The expansions of the above equations are, respectively,

$$f \sim -\frac{iQ}{4\beta}\left[1 + \frac{2i}{\pi}\left(\gamma + \ln\frac{k}{2}\right)\right] + \frac{1}{\beta} + \frac{Q}{2\pi\beta}\ln r + A\,(kr)^{1/2\beta} + \ldots \quad kr \to 0, \tag{4.3}$$

with $$A = \frac{4(\frac{1}{2})^{1/2\beta}}{\Gamma(1/2\beta)} \cos\frac{\omega}{2\beta} \cos\frac{\theta_0}{2\beta} . e^{-i\pi/4\beta}$$

$$f \sim \frac{iQ}{4(1-\beta)}\left[1 + \frac{2i}{\pi}\left(\gamma + \ln\frac{k}{2}\right)\right] - \frac{Q}{2\pi(1-\beta)}\ln r \quad kr \to 0. \tag{4.4}$$

For the inner problem we have the following two solutions.

(*a*) *Symmetrical case.* The generalization of equation (3.3) to the case of a gap of width d can be written

$$x + jy = \frac{dB(\beta)}{2} \exp\{\beta[q(f+jp)+s]\} + \ldots, \quad qf + s \gg 1, \tag{4.5}$$

for the ocean sector, with q and s constants to be determined according to a method used by Newman (1974).

For the sheltered sector of the plane we have

$$x + jy = -\frac{dB(\beta)}{2} \exp\{(\beta-1)[q(f+jp)+s]\} + \ldots, \quad qf + s \ll 1. \tag{4.6}$$

A factor $e^{-j\pi\beta}$ has been suppressed in the above expressions since the orientation of the two breakwaters plays no role in the matching of terms up to the order $(kr)^{1/2\beta}$.

Equations (4.5) and (4.6) give respectively

$$f \sim -\frac{1}{\beta q}\ln\frac{dB(\beta)}{2} - \frac{s}{q} + \frac{1}{\beta q}\ln r + \ldots \quad r \to \infty, \tag{4.7}$$

and

$$f \sim \frac{1}{(1-\beta)q}\ln\frac{dB(\beta)}{2}-\frac{s}{q}-\frac{1}{(1-\beta)q}\ln r+\dots \quad r\to\infty, \tag{4.8}$$

which are the generalizations to gap width d of results (3.4) and (3.5).

Matching the first few terms of equations (4.3) and (4.7) we get

$$-\frac{s}{q}-\frac{1}{\beta q}\ln\frac{dB(\beta)}{2}=\frac{1}{\beta}-\frac{iQ}{4\beta}\left[1+\frac{2i}{\pi}\left(\gamma+\ln\frac{k}{2}\right)\right];$$

$$\frac{1}{\beta q}=\frac{Q}{2\pi\beta}.$$

From this system of equations we obtain

$$Q=\frac{-2\pi}{\ln\dfrac{dB(\beta)}{2}+\beta s+\gamma+\ln\dfrac{k}{2}-\dfrac{i\pi}{2}} \quad\text{and}\quad q=\frac{2\pi}{Q}.$$

The other pair of equations (4.4) and (4.8) gives similarly

$$\frac{1}{(1-\beta)q}\ln\frac{dB(\beta)}{2}-\frac{s}{q}=\frac{iQ}{4(1-\beta)}\left[1+\frac{2i}{\pi}\left(\gamma+\ln\frac{k}{2}\right)\right],$$

from which, substituting the values of Q and q, we obtain

$$s=\frac{\ln\left[\tfrac{1}{2}dB(\beta)\right]+L}{1-\beta},$$

where $L=\gamma+\ln\tfrac{1}{2}k-\tfrac{1}{2}i\pi$.

Now Q and q can be written

$$Q=\frac{-2\pi(1-\beta)}{\ln\dfrac{dB(\beta)}{2}+L} \quad\text{and}\quad q=-s.$$

In the sheltered region the wave function becomes, from equation (4.2),

$$f \sim -\frac{\pi i}{2}H_0(kr).\frac{1}{\ln\left[\tfrac{1}{2}dB(\beta)\right]}+L, \tag{4.9}$$

whence $$|f|=\pi\left[\frac{J_0^2(kr)+Y_0^2(kr)}{4U^2+\pi^2}\right]^{\frac{1}{2}},$$

where $$U=\gamma+\ln\frac{kdB(\beta)}{4}$$

and J_0, Y_0 are the usual Bessel functions. The ratio of the wave height at a particular point for two values of β is therefore

$$\left(\frac{4U_2^2+\pi^2}{4U_1^2+\pi^2}\right)^{\frac{1}{2}}$$

with obvious notation.

(*b*) *Asymmetrical case.* For the ocean side the first term of equation (3.7) gives, after suppression of the factor $e^{2\pi j\beta}$,

$$w \sim (2\beta z)^{1/2\beta},$$

whence

$$F \sim -\frac{1}{4\pi\beta}\ln r + C - \frac{1}{4\pi\beta}\ln 2\beta. \tag{4.10}$$

In the sheltered region and for the same co-ordinate system as above, equation (3.10) gives similarly

$$w \sim (\alpha z)^{1/2(\beta-1)},$$

which leads to

$$F \sim \frac{1}{4\pi(1-\beta)}\ln r + C + \frac{1}{4\pi(1-\beta)}\ln \alpha. \tag{4.11}$$

If we put for convenience

$$C + \frac{1}{4\pi(1-\beta)}\ln \alpha = 0,$$

which is legitimate provided we keep C constant (Tuck 1971), we find from equations (4.10) and (4.11)

$$F \sim -\frac{1}{4\pi\beta}\ln r + \frac{V}{4\pi} \tag{4.12}$$

and

$$F \sim \frac{1}{4\pi(1-\beta)}\ln r, \tag{4.13}$$

where

$$V = \frac{\ln[d\beta^{2\beta}(1-\beta)^{2(1-\beta)}]}{\beta(1-\beta)}.$$

Putting as previously $F = qf + s$ and performing a matching between the two pairs of equations (4.3), (4.12) and (4.4), (4.13), we obtain the relations

$$-\frac{1}{4\pi\beta}\cdot\frac{1}{q} = \frac{1}{2\pi}\cdot\frac{Q}{\beta}, \quad \frac{1}{4\pi(1-\beta)}\cdot\frac{1}{q} = -\frac{1}{2\pi}\cdot\frac{Q}{1-\beta},$$

$$\frac{V}{4\pi q} - \frac{s}{q} = \frac{QL}{2\pi\beta} + \frac{1}{\beta}, \quad -\frac{s}{q} = -\frac{QL}{2\pi(1-\beta)}.$$

The excessive equation confirms the correct choice of the function multiplying Q in the outer expansions. The solution of the above simultaneous equations is

$$q = -\frac{1}{2Q}, \quad Q = -\frac{2\pi(1-\beta)}{\beta(1-\beta)V + L}, \quad s = -L/4\pi(1-\beta).$$

In the protected region we get from equation (4.2)

$$f \sim -\frac{i\pi H_0(kr)}{2}\cdot\frac{1}{\beta(1-\beta)V + L} \tag{4.14}$$

on the assumption that the incident wave is of unit amplitude. The wave height obtained from equation (4.14) is

$$\pi\left[\frac{J_0^2(kr) + Y_0^2(kr)}{4W^2 + \pi^2}\right]^{\frac{1}{2}},$$

where $W = \gamma + \ln\frac{1}{2}k + \beta(1-\beta)V$.

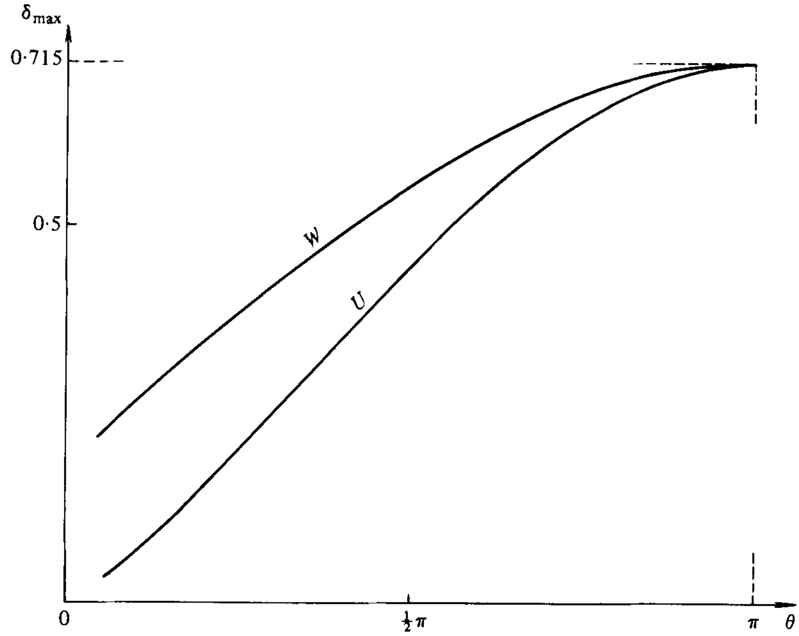

FIGURE 4. Range of applicability of the theory. Lower line refers to the symmetrical opening.

The ratio of the wave heights for two values of θ is now

$$\frac{|f_1|}{|f_2|} = \left(\frac{4W_2^2+\pi^2}{4W_1^2+\pi^2}\right)^{\frac{1}{2}}. \tag{4.15}$$

5. Evaluation of the results

In the range of practical values of $\beta(\frac{1}{2} < \beta < \frac{3}{4})$ it can be seen from equation (4.14) that, for any two values $\beta_1 > \beta_2$, we have $|f_1| > |f_2|$, which gives $|W_2| > |W_1|$ through equation (4.15). Considering now the function $\ln[\beta^\beta(1-\beta)^{(1-\beta)}]$ and noting that $\delta\,(=d/\lambda) < 1$, we conclude that W is a non-positive quantity in the above range of β. The maximum acceptable gap width happens therefore at $W = 0$; this gives a δ_{max} of 0·715 occurring at $\theta = \pi$.

Similar considerations lead to corresponding restrictions as regards the gap width in the symmetrical case. Both ranges of applicability of this theory are shown in figure 4 with respect to maximum gap width for $0 < \theta < \pi$.

The values of δ_{max} for $\theta = \frac{1}{2}\pi$ are 0·550 and 0·443 for the asymmetrical and symmetrical case respectively. Generally the theory can be applied to the former case for gap widths greater than for the latter.

A quantity is introduced now related to the transmission of energy through the passage as follows:

$$T = \frac{1}{H_{in}^2 d}\int_0^\theta H^2(r,\omega)\,r\,d\omega,$$

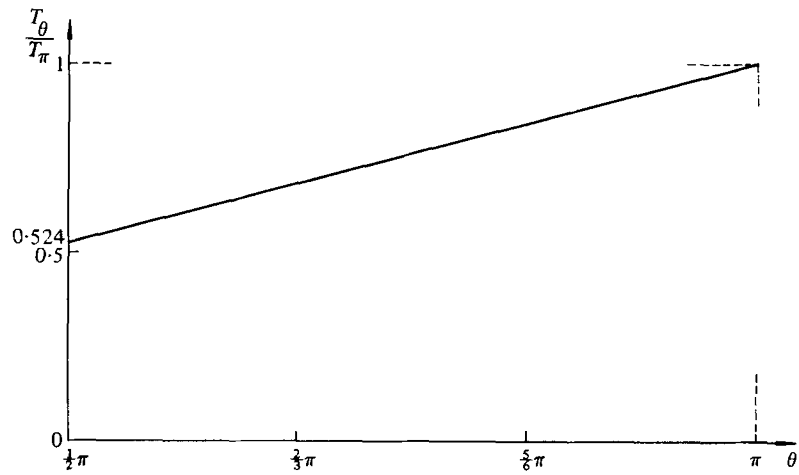

FIGURE 5. The ratio of the transmission factors T_θ , T_π as a function of θ.

in which H_{in} is the height of the incident wave and $H(r, \omega)$ the wave height at the point (r, ω). This relation can be approximated by

$$T = \frac{\rho}{\delta}\sum_i K_i^2 \theta_i, \tag{5.1}$$

where $\rho = r/d$ and K_i is the diffraction coefficient at the point $P_i(\rho, \omega)$. If K_i is assumed uniform along the arc ρ = constant, equation (5.1) yields

$$T = \frac{\theta\rho}{\delta}|f|^2. \tag{5.2}$$

Equations (4.9) and (4.14) combined with (5.2) and the expansion

$$J_0^2(kr) + Y_0^2(kr) \sim \frac{1}{\pi^2\rho} - \dots$$

for large kr, give for the symmetrical and asymmetrical opening, respectively,

$$T = \frac{\theta}{\delta}\cdot\frac{1}{4U^2+\pi^2}, \quad T = \frac{\theta}{\delta}\cdot\frac{1}{4W^2+\pi^2}. \tag{5.3}$$

The ratio of the transmission coefficient T associated with an angle θ, to the corresponding one of $\theta = \pi$ has been drawn in figure 5 as a function of θ for the asymmetrical opening.

It is noted that the relation shown in this figure is close to a linear dependence, with maximum deviation of less than 5%.

The transmission coefficient of equations (5.3) expresses the proportion of the energy incident on the gap that penetrates into the protected area. For small enough gaps this definition of T results in transmission coefficient greater than one. It is evident that the incident energy changes with the gap width. However, it is possible to relate T to a constant incident energy. The energy transmitted for $\delta_{\max}$ is taken as equal to one, with which the other coefficients are compared. The corresponding trans-